LINEAR ALGEBRA I - SEMESTRAL EXAM. MAX MARKS : 60

Answer all questions. You may use results proved in class after correctly quoting them.

(1) (a) Solve the system of equations

$$\begin{aligned} y - z - 2w &= 0 \\ 2x - 3y - 3z + 6w &= 2 \\ 4x + y + z - 2w &= 4. \end{aligned}$$

[6]

(b) Let A be a 4×3 matrix with real entries. Let B be the 8×6 block matrix

$$B = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

What is the row echelon form of the matrix B? [2]

(c) Let A be a matrix such that $A^3 - 4A^2 + 3A - 5I = 0$. Must A be invertible? [2]

(2) (a) Let B be the matrix

$$B = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 \\ 1 & 2 & -2 & 1 \end{pmatrix}$$

Exhibit a basis for the row space and a basis of the column space of B. Justify. [9]

(b) A real matrix B is said to be skew-symmetric if $B = -B^t$. Show that if B is a $n \times n$ skew-symmetric matrix, then $(I - B)$ has rank n. [9]

(3) (a) Let

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix}$$

be the matrix of a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ relative to the basis $E = (e_1, e_2, e_3)$. Find the matrix of T relative to the basis $B = (e_1 + e_2, e_2 + e_3, e_3 + e_1)$. [6]

(b) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by $T(x, y)^t = (x - y, x + y, y)^t$. Show that T is linear. Find a matrix A such that $T(X) = AX$ for every $X \in \mathbb{R}^2$. Describe the kernel and image of T. Find $\dim(\ker(T))$ and $\operatorname{rank}(T)$. [8]

(c) Let P denote the real vector space of polynomials in x of degree at most four with real coefficients. Let A be the set

$$A = \{(a, b) : \text{there is a linear map } T : P \longrightarrow \mathbb{R}^6 \text{ with } \operatorname{rank}(T) = a, \dim(\ker(T)) = b\}.$$

Determine the number of elements in A. Justify. [8]

(4) What do you mean by a generalized inverse of a matrix? Find a generalized inverse G of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

and hence describe (using G) the complete set of solutions of the system [2+4+4]

$$AX = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$